Cor 2.4 Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$, and $V = \{(x, y) \in \mathbb{R}^2 \,|\, z = x + iy \in U\}$. Then f is analytic in U if and only if u and v are continuously differentiable in V and if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Def 4.2 A harmonic function on $V \subset \mathbb{R}^2$ is a twice continuously differentiable function $u : V \to \mathbb{R}$ such that the Laplace's equation holds

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Thm 4.5 Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be analytic, where $f = u + iv$. Then u, v are harmonic functions

Cor 8.4 $U \subset \mathbb{C}$ open $f : U \to \mathbb{C}$ analytic then f is C^∞

Thm 8.5(Liouville) Every bounded entire function is constant

Thm 7.7[Maximums modolus principle] $K \subset \mathbb{C}$ compact, and $f : K \to \mathbb{C}$ continuous and analytic on $U = Int(K)$ interior of K. Then, $|f| : K \to \mathbb{C}$, $z \mapsto |f(z)|$ assumes its maximum on ∂K.

Def The concept is the same as the principal branch but, $Log(w) = ln|w| + iarg\, w + i2k\pi$ where $k = 0, \pm 1, \pm 2, \ldots.$

Sin and Cos

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos(z) = \frac{e^{iz} - e^{-iz}}{2}$$

$$e^{iz} = \sum_{n=0}^{\infty} \frac{i^n}{n!} z^n, \ \sin(z) = \sum_{n \geq 0} \frac{i^{2n}}{(2n+1)!} z^{2n+1}$$

$$\cos(z) = \sum_{n \geq 0} \frac{i^{2n}}{(2n)!} z^{2n}$$

Series

Thm (Cauchy-Hadamard) Let a power series $\sum_{n=0}^{\infty} a_n z^n$, then the radius of convergence is given by $R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}}$ Moreover, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely if $|z| < R$ (diverges otherwise).

Part. Frac g, h complex polynomials, $h = c \cdot \prod_{i=1}^{s} (z - z_i)^{n_i} \neq 0$ with $z_1, \ldots, z_s$ pairwise distinct and $n_i, \ldots, n_s \geq 1$. Then there are a polynomial q and $c_{i,-n_i}, \ldots, c_{i,-1} \in \mathbb{C}$ for $i = 1, \ldots, s$ such that

$$\frac{g}{h} = q + \sum_{i=1}^{s} \sum_{k=-n_i}^{-1} c_{i,k} (z - z_i)^k$$

Complex integration

Length Let γ be a smooth arc with parametrization $z : [a, b] \to \mathbb{C}$. The length of γ is

$$l(\gamma) = \int_a^b |z'(t)| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt$$

Def $U \subset \mathbb{C}$ open, $f : U \to \mathbb{C}$ continuous, $f = u + iv$ and $z : [a, b] \to \mathbb{C}$ smooth arc with $\gamma = \operatorname{Im}(z) \subset U$, $z = x + iy$. Then the integral of f along γ is

$$\int_\gamma f = \int_\gamma f(z) dz = \int_a^b f(z(t)) z'(t) dt$$

Thm 5.5 $f : U \to \mathbb{C}$ continuous with primitive F. $\Gamma \subset U$ contour with parametrization $z : [a, b] \to U$ then

$$\int_\Gamma f = F(z(b)) - F(z(a))$$

Pro 5.7 $U \subset \mathbb{C}$ open, $f : U \to \mathbb{C}$ continuous, $\Gamma \subset U$ contour of legth $l(\Gamma)$ and $M = \max\{|f(z)| \,|\, z \in \Gamma\}$ then

$$\left| \int_\Gamma f \right| \leq M l(\gamma)$$

Thm 6.1(Cauchy) Let $U \subset \mathbb{C}$ open, $f : U \to \mathbb{C}$ analytic and $\Gamma \subset U$ closed contour. If Γ is contractible, then

$$\int_\Gamma f(z) dz = 0$$

Thm(Cauchy Integral formula) Let Γ be a simple closed positively oriented contour. If f is analytic in some simply connected domain D containing Γ and z_0 is any inside Γ, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Thm Let $U \subset \mathbb{C}$ open with $\mathbb{R} \subset U$, and $S = \{z \in \mathbb{C} \backslash U \,|\, \operatorname{Im} z > 0\}$. Take $f : U \to \mathbb{C}$ analytic such that $S \subset$ {poles of f}. If there are $c, r > 0$ such that $|f(x)| \leq \frac{c}{|x|^2}$ for all $z \in U$ with $\operatorname{Im} z \geq 0$, then the limit converges, S is finite and

$$\int_{-\infty}^{\infty} f(x) dx = 2\pi i \cdot \sum_{z \in S} \operatorname{Res}_z f$$

Thm 11.7 g, h polynomials with $deg\, h \geq deg\, g + 1$, $h(z) \neq 0$ for $x \in \mathbb{R}$, $k \neq 0$ then

$$\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} e^{ikx} dx = 2\pi i \sum_{\substack{h(z_0)=0 \\ k \cdot Im(z_0) > 0}} \operatorname{Res}_{z_0} \left(\frac{g(z)}{h(z)} e^{ikz} \right)$$

Residues & Zeros

Cor $f : U \to \mathbb{C}$ analytic and z_0 pole of f of order $m \geq 1$. Then,

$$\operatorname{Res}_{z_0} f = \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right]_{z=z_0}$$

Cor $f : U \to \mathbb{C}$ analytic with a simple pole at z_0. Let $g : U \cup \{z_0\} \to \mathbb{C}$ analytic. Then

$$\operatorname{Res}_{z_0} fg = g(z_0) \cdot \operatorname{Res}_{z_0} f$$

Def $f : U \to \mathbb{C}$ analytic and $S = \{z \in U \,|\, f(z) = 0\}$ zeros of f. The logarithmic derivative of f is $\frac{f'}{f} : U \backslash S \to \mathbb{C}$

Pro $f : D_\epsilon^\circ \subset U \to \mathbb{C}$ with $\operatorname{ord}_{z_0} f \neq -\infty$ and $f(z) \neq 0$ for $z \in D_\epsilon^\circ(z_0)$. Then

- $\operatorname{ord}_{z_0} \frac{f'}{f} = -1$ if $\operatorname{ord}_{z_0} f \neq 0$
- $\operatorname{ord}_{z_0} \frac{f'}{f} \geq 0$ if $\operatorname{ord}_{z_0} f = 0$
- $\operatorname{Res}_{z_0} \frac{f'}{f} = \operatorname{ord}_{z_0}(f)$

Def(Winding Number) Let γ a closed arc in $\mathbb{C}$, and assume $z \in \mathbb{C} \backslash \gamma$. The **winding number** of γ around z_0 is

$$W(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z} dw$$

Thm (Residue Formula) Let $U \subset \mathbb{C}$ be open, and $\gamma \subset U$ be a contractible closed arc. Assume $f : U \backslash \{z_1, \ldots, z_s\} \to \mathbb{C}$ analytic, where $\{z_1, \ldots, z_s\} \in U \backslash \gamma$ are poles of f. Then,

$$\int_\gamma f = 2\pi i \sum_{i=1}^{s} W(\gamma, s_i) \cdot \operatorname{Res}_{z_i} f$$

Thm 12.2 (Argument principle) $f : U \backslash \{z_1, \ldots, z_s\} \to \mathbb{C}$ analytic with poles $\{z_1, \ldots, z_s\}$, $\gamma \subset U \backslash \{z_1, \ldots, z_s\}$ boundary curve such that $f(z) \neq 0$ for $z \in \gamma$ then

$$\sum_{z \in \gamma^{int}} \operatorname{ord}_z f = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz$$

Thm 12.4 (Rouche) $f, g : U \to \mathbb{C}$, U simply connected, $\gamma \in U$ boundary curve with $\gamma^{int} \subset U$ if $|f(z) - g(z)| < |g(z)|$ $\forall z \in \gamma$, then

$$\sum_{z \in \gamma^{int}} \operatorname{ord}_z f = \sum_{z \in \gamma^{int}} \operatorname{ord}_z g$$

Thm 12.7 $\mathcal{F}(\hat{\mathbb{C}})\{g/h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}$ and for every $f \in \mathcal{F}(\hat{\mathbb{C}}) \backslash \{0\}$, $\sum_{z \in \hat{\mathbb{C}}} \operatorname{ord}_z f = 0$